AN INTRODUCTION TO K-THEORY

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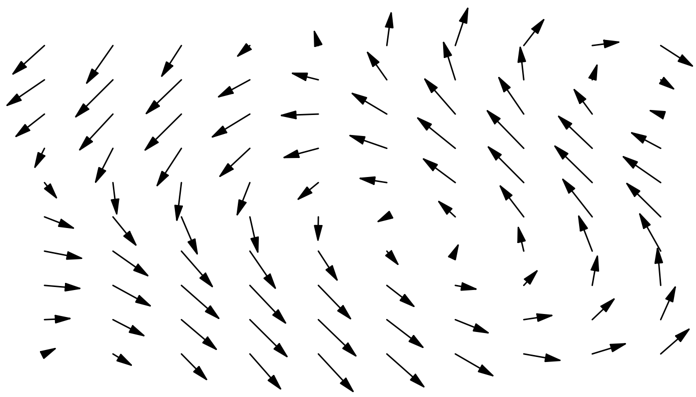
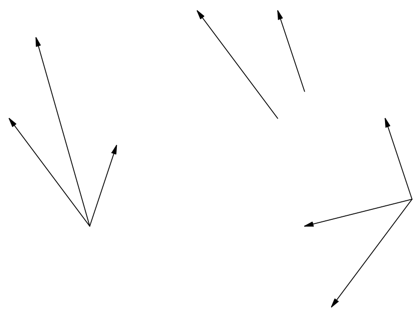
1. Vector bundles

A vector bundle over a space involves specifying a vector space at each point. When we study differential equations: we choose a tangent vector at each point:



Here each vector must belong to some vector space. The collection of all these vector spaces makes up a vector bundle. And K-theory is defined by considering all possible vector bundles.

A key point: We can take linear combinations of vectors based at the same point, but not of vectors based at different points; they belong to separate vector spaces:



These would be real vector spaces, but we're going to work, for simplicity, uniquely with the complex case.

Though the vector spaces are different at different points, we want them to be not *too* different if the points are close, i.e., we want a continuity condition.

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For bundles we do this in a very strong way: *local triviality*. We insist that in some neighbourhood U of each point, the bundle looks just like a product $U \times \mathbb{C}^n$.

Now we need to turn these ideas into a formal definition. We will do (almost) everything over a compact Hausdorff space, and often it will be connected too.

Definition 1. An n-dimensional *vector bundle* over X is a continuous map $p : E \to X$ such that $E_x := p^{-1}(x)$ is an n-dimensional vector space over $\mathbb{C}$, and X is covered by a collection of open sets U such that there is a map $\varphi : p^{-1}(U) \to \mathbb{C}^n$ for which

(1) $e \mapsto \bigl(p(e), \varphi(e)\bigr)$ is a homeomorphism $p^{-1}(U) \to U \times \mathbb{C}^n$,
(2) $\varphi\big|_{E_x}$ is a linear isomorphism for each $x \in U$.

Think of E as the set of all possible arrows, and p as the function giving the base of the arrow. So E_x is the set of arrows based at x.

Definition 2. A *homomorphism* $\alpha : E \to F$ between vector bundles over X is a continuous map such that $p_F \circ \alpha = p_E$, and that for each $x \in X$ the restricted map $E_x \to F_x$ is linear. This means that if both bundles are trivial over an open set U, then α determines a function from U to a set of matrices. Actually we need to specify the trivialising functions (the φ's in Definition 1) to get the matrices, just as we must to specify bases to get a matrix from a linear transformation.

Definition 3. We let $\mathrm{Vect}_n(X)$ denote the category in which the objects are n-dimensional vector bundles over X and the morphisms are vector bundle homomorphisms.

2. Vect_n is a Functor

We can think of Vect_n as a contravariant functor $\mathbf{Top} \to \mathbf{Cat}$.

Given $f : X \to Y$ in $\mathbf{Top}$, we define $f^* : \mathrm{Vect}_n(Y) \to \mathrm{Vect}_n(X)$ by sending $E \in \mathrm{Vect}_n(Y)$ to

$$f^*E = \bigl\{\, (x, e) \in X \times E : f(x) = p(e) \,\bigr\},$$

with $p : f^*E \to X$ sending (x, e) to x. We see that $(f^*E)_x = \{x\} \times E_{f(x)}$, so we give it the vector space structure of $E_{f(x)}$. If $\varphi : p^{-1}(V) \to \mathbb{C}^n$ defines the triviality of E over $V \subset Y$, then

$$\begin{aligned} p^{-1}\bigl(f^{-1}(V)\bigr) = \bigl\{\, (x, e) : f(x) = p(e) \in V \,\bigr\} &\to \mathbb{C}^n \\ (x, e) &\mapsto \varphi(e) \end{aligned}$$

establishes triviality of f^*E over $f^{-1}(V) \subset X$.

Thus $f^*E \in \mathrm{Vect}_n(X)$.

It is easy to work out how a homomorphism $\alpha : E \to F$ between bundles over Y gives rise to $f^*\alpha : f^*E \to f^*F$ in $\mathrm{Vect}_n(X)$. Checking that $f^*\alpha :$

$\mathrm{Vect}_n(Y) \to \mathrm{Vect}_n(X)$ is a (covariant) functor, i.e., a morphism in the category of categories, is routine.

Actually we should probably be working with *graded* categories here, and allow linear maps between bundles of different dimensions, but this doesn't actually add much.

If $i : A \subset X$ is the inclusion and $E \in \mathrm{Vect}_n(X)$, we often write $E\big|_A$ for $i^*E \in \mathrm{Vect}_n(A)$.

3. Vect_n is a Homotopy Functor

In some ways this is the crucial observation, where we move from the topological world (bundles are defined in terms of homeomorphisms) to the homotopy world. It is therefore worth spending some time seeing how this works.

We will assume that X is *compact* and *Hausdorff*.[1]

For any $t \in [0,1]$, let $\alpha_t : X \to X \times [0,1]$ be the map $x \mapsto (x,t)$.

Proposition 4. *If E is a vector bundle over $X \times [0,1]$, then α_0^*E and α_1^*E are isomorphic.*

Proof. We need to use the

Bundle Creeping Lemma (BCL). *For any $t \in [0,1]$, there exists $\delta > 0$ such that α_u^*E is isomorphic to α_t^*E for all $u \in (t-\delta, t+\delta)$.*[2]

The proof of the proposition now follows closely those of basic results in real analysis.

Let $K := \{\, u \in [0,1] : \alpha_0^*E \cong \alpha_u^*E \,\}$ (clearly non-empty and bounded above), and let $s := \sup K$. Use the BCL to show that $s \in K$, and then use it again to show that $s < 1$ is impossible.

So the real work lies in the

Proof of the BCL.

Let $\pi : X \times [0,1] \to X$ be $\pi(x,u) = x$, and form the bundle $F := \pi^*\alpha_t^*E \in \mathrm{Vect}_n\big(X \times [0,1]\big)$.

A Simple Fact: $\alpha_u^*F = \alpha_t^*E$ for all $u \in [0,1]$ since $\pi \circ \alpha_u = 1_X$.

Now α_t is just the inclusion

$$X \to X_t := X \times \{t\} \subset X \times [0,1],$$

so the particular case $u = t$ of the simple fact shows that there is an isomorphism[3] defined between E and F over X_t.

We are going to extend this isomorphism

$$E\big|_{X_t} \to F\big|_{X_t}$$

[1] The results hold with weaker assumptions, but we are trying to simplify things.

[2] We ignore here, and in the proof of the BCL, the minor modifications necessary when $t = 0$ or $t = 1$.

[3] It's actually the identity.

to an isomorphism

$$E\big|_{X\times(t-\delta,t+\delta)} \to F\big|_{X\times(t-\delta,t+\delta)}$$

for some $\delta > 0$. The simple fact shows that this will prove the BCL.

Extending the isomorphism to a linear map locally, i.e., in a neighbourhood of (x,t) is not a problem. For local triviality says that we just need to extend a matrix valued function, and the Tietze Extension Theorem (several complex variable case) says that can be done.[4] In fact the definition of the product topology tells us that we can assume that we've extended the homomorphism over a set of the form $U_x \times (t-\delta_x, t+\delta_x)$ for some neighbourhood U_x of x in X and some $\delta_x > 0$.

By compactness we can pick just a finite number of these open sets U_x, and then take $\tilde{\delta}$ just less than the minimum of the corresponding δ_x's to find open sets $U_1, \ldots, U_k$ covering X such that the isomorphism between $E\big|_{X_t}$ and $F\big|_{X_t}$ extends over each $U_j \times [t-\tilde{\delta}, t+\tilde{\delta}]$.

Now we choose[5] a "partition of unity": a set of continuous functions $\lambda_j : X \times [t-\tilde{\delta}, t+\tilde{\delta}] \to \mathbb{R}$ such that

(1) $\lambda_j(x,u) = 0$ if $x \notin U_j$,
(2) $\sum_{j=1}^{k} \lambda_j(x,u) = 1$ for all (x,u).

If

$$\alpha_j : E\big|_{U_j\times[t-\tilde{\delta},t+\tilde{\delta}]} \to F\big|_{U_j\times[t-\tilde{\delta},t+\tilde{\delta}]}$$

is our extension, we define

$$\beta_j : E\big|_{X\times[t-\tilde{\delta},t+\tilde{\delta}]} \to F\big|_{X\times[t-\tilde{\delta},t+\tilde{\delta}]}$$

by letting β_j agree with the scalar multiple $\lambda_j\alpha_j$ over $U_j \times [t-\tilde{\delta}, t+\tilde{\delta}]$, but be the zero homomorphism outside this (relatively) open set. The support condition, and the continuity of λ_j, ensure that β_j is a vector bundle homomorphism. And the addition property means that

$$\beta := \sum_{j=1}^{k} \beta_j$$

agrees over X_t with the isomorphism we started with.[6]

The only remaining problem is that we extended an *isomorphism* to a *homomorphism*—all these zero homomorphisms seem to make invertibility problematic.

However, locally we just have a continuous family of matrices, and invertibility is equivalent to the determinant being non-zero. So continuity implies[7] that there must be a $0 < \delta \le \tilde{\delta}$ such that β is invertible over $X \times (t-\delta, t+\delta)$. $\square$

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[4] We need X normal here, which it is. See [6, 6.**O**] for the Tietze Extension Theorem

[5] Again, we can do this since X is normal; see [6, 5.**W**].

[6] Yes, we *can* add homomorphisms.

[7] A continuous function non-zero at a point is non-zero in a neighbourhood of that point.

The hard work is now done, and it follows quite simply that:

(1) If $f, g : X \to Y$ are homotopic and $E \in \mathrm{Vect}_n(Y)$, then f^*E and g^*E are isomorphic.
(2) If X and Y are homotopy equivalent, then $\mathrm{Vect}_n(X)$ and $\mathrm{Vect}_n(Y)$ are equivalent categories.
(3) If X is contractible, then any vector bundle over X is trivial, i.e., isomorphic to $X \times \mathbb{C}^n$.

4. Adding and Multiplying

There are natural transformations:

Direct sum:

$$\begin{aligned} \mathrm{Vect}_m(X) \times \mathrm{Vect}_n(Y) &\to \mathrm{Vect}_{m+n}(X \times Y) \\ (E, F) &\mapsto E \boxplus F, \end{aligned}$$

where $E \boxplus F = E \times F$ as a topological space with $p : E \boxplus F \to X \times Y$ sending (e, f) to $\big(p(e), p(f)\big)$. Thus $(E \boxplus F)_{(x,y)} = E_x \times F_y$, which is given the vector space structure $E_x \oplus F_y$.

Tensor product:

$$\begin{aligned} \mathrm{Vect}_m(X) \times \mathrm{Vect}_n(Y) &\to \mathrm{Vect}_{mn}(X \times Y) \\ (E, F) &\mapsto E \boxtimes F, \end{aligned}$$

which is defined so that $(E \boxtimes F)_{(x,y)} = E_x \otimes F_y$.

Letting $Y = X$ and following with Δ^*, where $\Delta : X \to X \times X$ is the diagonal, gives *internal* operations

$$\begin{aligned} \mathrm{Vect}_m(X) \times \mathrm{Vect}_n(X) &\to \mathrm{Vect}_{m+n}(X) \\ (E, F) &\mapsto E \oplus F, \end{aligned}$$

with $(E \oplus F)_x = E_x \oplus F_x$, and

$$\begin{aligned} \mathrm{Vect}_m(X) \times \mathrm{Vect}_n(X) &\to \mathrm{Vect}_{mn}(X) \\ (E, F) &\mapsto E \otimes F, \end{aligned}$$

with $(E \otimes F)_x = E_x \otimes F_x$.

There are lots of natural isomorphisms around here. For example, if we have three bundles $E \in \mathrm{Vect}_m(X)$, $F \in \mathrm{Vect}_n(Y)$ and $G \in \mathrm{Vect}_p(Z)$, then $E \boxtimes (F \boxplus G)$ is naturally isomorphic to $\tilde{\Delta}^*\big((E \boxtimes F) \boxplus (E \boxtimes G)\big)$, where $\tilde{\Delta} : X \times Y \times Z \to X \times Y \times X \times Z$ sends (x, y, z) to (x, y, x, z).

If we were going to be serious about this, we would have to start talking about symmetric monoidal categories at this point. But for us, this is just a staging post on our journey.

5. K-theory

All of this is structurally rather nice, and geometrically promising because

(1) vector bundles occur in nature (tangent bundles of manifolds and other bundles derived from them);
(2) vector bundles allow easy geometrical constructions. For example if $f : X \to Y$ is a *finite covering*, and $E \in \mathrm{Vect}_n(X)$, then we can define $f_*E \in \mathrm{Vect}_n(Y)$ as the bundle such that

$$(f_*E)_y = \bigoplus_{x \in f^{-1}(y)} E_x.$$

On the other hand in algebraic topology we like to have invariants arriving in familiar categories (groups, rings, etc.) in which we can hope to detect things. The category of categories, with direct sum and tensor product, is rather too complicated for comfort.

What can we do?

Firstly we can consider *isomorphism classes* within $\mathrm{Vect}_n(X)$. We will let $\overline{\mathrm{Vect}}_n(X)$ denote this set of isomorphism classes.[8] If $E \in \mathrm{Vect}_n(X)$, it is usual to write $[E] \in \overline{\mathrm{Vect}}_n(X)$ for the isomorphism class containing E. Direct sum and tensor product now give us operations

$$\begin{aligned} \overline{\mathrm{Vect}}_m(X) \times \overline{\mathrm{Vect}}_n(X) &\to \overline{\mathrm{Vect}}_{m+n}(X) \\ \big([E],[F]\big) &\mapsto [E \oplus F], \end{aligned}$$

and

$$\begin{aligned} \overline{\mathrm{Vect}}_m(X) \times \overline{\mathrm{Vect}}_n(X) &\to \overline{\mathrm{Vect}}_{mn}(X) \\ \big([E],[F]\big) &\mapsto [E \otimes F]. \end{aligned}$$

Now $\overline{\mathrm{Vect}}_*$ is a functor from **Top**, or rather **hTop**, to the category of semi-rings.

But semi-rings aren't very tractable, so we perform an algebraic construction and consider *formal differences* $[E] - [F]$ of elements of $\overline{\mathrm{Vect}}_*(X)$, subject to relations like $[E \oplus G] - [F \oplus G] = [E] - [F]$. This is just like $\mathbb{Z}$ is constructed from $\mathbb{N}$, or, multiplicatively, like the way that the field of fractions of an integral domain is built.

The result is the ring $K(X)$.

The general construction:

$$\text{category with a sum} \to \text{group}$$

was invented by Grothendieck; the K probably stands for *Klasse*.

[8] There is a conflict of notation with most texts here.

6. Example: the 2-sphere

We will consider the 2-sphere as the *Riemann sphere* $\mathbb{C} \cup \{\infty\}$, or more formally as

$$\begin{aligned} CP^1 &= \big(\mathbb{C}^2 \smallsetminus \{(0,0)\}\big)/\mathbb{C}^\times \\ &= \big\{\, [z_0, z_1] : z_0, z_1 \in \mathbb{C},\ \text{not both zero} \,\big\}, \end{aligned}$$

using *projective coordinates*, so that $[z_0, z_1] = [\lambda z_0, \lambda z_1]$ for any $\lambda \in \mathbb{C}^\times = \mathbb{C} \smallsetminus \{0\}$.

We define the one-dimensional bundle L (a line bundle) over CP^1 as a sub-bundle of the trivial two-dimensional bundle $CP^1 \times \mathbb{C}^2$:

$$L := \big\{\, \big([z_0, z_1], (\lambda z_0, \lambda z_1)\big) : \lambda \in \mathbb{C} \,\big\}.$$

We see that $L_{[z_0,z_1]}$ is indeed a subspace of $\big\{[z_0, z_1]\big\} \times \mathbb{C}^2$ which doesn't depend on the choice of projective coordinates. For local triviality we will show that $L\big|_{U_0}$ is trivial, where

$$U_0 := \big\{\, [1, z] \in CP^1 \,\big\} = CP^1 \smallsetminus \big\{[0,1]\big\},$$

and similarly for

$$U_1 := \big\{\, [w, 1] \in CP^1 \,\big\} = CP^1 \smallsetminus \big\{[1,0]\big\}.$$

To see this define $\varphi : L\big|_{U_0} \to \mathbb{C}$ by

$$\big([1, z], (\lambda, \lambda z)\big) \mapsto \lambda.$$

This is certainly a linear isomorphism on fibres, and

$$\big([1, z], (\lambda, \lambda z)\big) \mapsto \big([1, z], \lambda\big)$$

is certainly a homeomorphism.

Similarly $\varphi : L\big|_{U_1} \to \mathbb{C}$ is given by

$$\big([w, 1], (\mu w, \mu)\big) \mapsto \mu.$$

Composing the inverse of one homeomorphism with the other, we get, if $z \neq 0$,

$$\big([1, z], \lambda\big) \mapsto \big([1, z], (\lambda, \lambda z)\big) = \big([z^{-1}, 1], (\lambda, \lambda z)\big) \mapsto \big([z^{-1}, 1], \lambda z\big).$$

What's happening here is that both U_0 and U_1 are homeomorphic to $\mathbb{C}$. They are glued together in CP^1 via $z \mapsto z^{-1}$, but the trivial line bundles over them are identified by the linear isomorphism given, at the point z, by multiplication by z.

The isomorphism is determined by the identity function

$$\mathbb{C}^\times \to GL(1, \mathbb{C}) = \mathbb{C}^\times.$$

(It's all so tautological!) This is not homotopically trivial, so L is not a trivial bundle.

In fact we can construct *all* bundles on S^2 in the following way:

Take the North and South hemispheres of $S^2 = CP^1$. Each hemisphere is contractible, so we have no choice but to take a trivial bundle over each of them. Now glue the two bundles over the equator. This means we need to choose a vector bundle isomorphism

$$S^1 \times \mathbb{C}^n \to S^1 \times \mathbb{C}^n,$$

in other words a continuous map $S^1 \to GL(n, \mathbb{C})$.

This is an example of the *clutching construction*. It's not too hard to see that local triviality is retained, and that the resulting isomorphism class depends only on the homotopy class of the map.

Hence $\overline{\mathrm{Vect}}_n(S^2) = [S^1, GL(n, \mathbb{C})] = \pi_1\big(GL(n, \mathbb{C})\big)$.

Where does the bundle L fit in?

Well, the southern hemisphere is

$$\big\{\, [1, z] \in CP^1 : |z| \leq 1 \,\big\} \subset U_0,$$

and the northern hemisphere is

$$\big\{\, [w, 1] \in CP^1 : |w| \leq 1 \,\big\} \subset U_1.$$

So the intersection is the set where $|z| = |w| = 1$. But note that $[1, z] = [z^{-1}, 1]$, so the identity map on the equator is given by $w = z^{-1}$. The calculations we did above showed that the map $S^1 \to \mathbb{C}^\times = GL(1, \mathbb{C})$ is the inclusion $z \mapsto z$.

Now this is a generator of $\pi_1\big(GL(1, \mathbb{C})\big) = \mathbb{Z}$. Applying the clutching construction with $z \mapsto z^k$ for $k \in \mathbb{Z}$, defines a bundle which we denote by L^k. If $k > 0$, this is the tensor power

$$L^k = L^{\otimes k} = \underbrace{L \otimes L \otimes \cdots \otimes L}_{k}.$$

These are all the line bundles over S^2, up to isomorphism.

In fact $\overline{\mathrm{Vect}}_n(S^2) = \pi_1\big(GL(1, \mathbb{C})\big) = \pi_1\big(U(n)\big) = \mathbb{Z}$ for all n, because of the following diagram

$$\begin{array}{ccccccccc}
1 & \longrightarrow & SL(n, \mathbb{C}) & \longrightarrow & GL(n, \mathbb{C}) & \xrightarrow{\det} & GL(1, \mathbb{C}) & \longrightarrow & 1 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
1 & \longrightarrow & SU(n) & \longrightarrow & U(n) & \xrightarrow{\det} & U(1) & \longrightarrow & 1
\end{array}$$

in which $SL(n, \mathbb{C})$ and $U(n)$ are simply-connected and $GL(n, \mathbb{C})/U(n)$ is contractible.[9]

So we can detect the homotopy class of maps $S^1 \to GL(n, \mathbb{C})$ by taking the determinant. It follows, for example, that the two clutching maps $S^1 \to$

[9]We prefer $U(n)$ because it is compact. There is another way to think of the restriction to $U(n)$: put a hermitian metric on each fibre, then insist that the clutching function is unitary.

$GL(2,\mathbb{C})$ given by

$$z \mapsto \begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix}$$

and

$$z \mapsto \begin{pmatrix} z^2 & 0 \\ 0 & 1 \end{pmatrix}$$

are homotopic,[10] and so they must yield isomorphic bundles over S^2, for the determinant of each map is $z \mapsto z^2$.

In other words, we see that $L \oplus L$ and $L^2 \oplus 1$ are isomorphic in $\mathrm{Vect}_2(S^2)$. Here $1 = L^0$ is the trivial line bundle.

We now know all there is to know about $\overline{\mathrm{Vect}}_n(S^2)$. Write n for the trivial n-dimensional bundle, and let $E(k,n)$ denote the bundle $L^k \oplus (n-1)$ represented by the clutching map

$$z \mapsto \begin{pmatrix} z^k & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix},$$

then

$$\begin{aligned} \overline{\mathrm{Vect}}_0(S^2) &= \{\,[0]\,\}, \\ \overline{\mathrm{Vect}}_n(S^2) &= \big\{\,\big[E(k,n)\big] : k \in \mathbb{Z}\,\big\}, \qquad \text{for } n > 0. \end{aligned}$$

Simple matrix calculations show that these bundles add and multiply as follows:

$$\begin{aligned} E(j,m) \oplus E(k,n) &\cong E(j+k, m+n), \\ E(j,m) \otimes E(k,n) &\cong E(jn+km, mn). \end{aligned}$$

Now to form $K(S^2)$ we need to introduce "virtual bundles", of non-positive dimension. Let

$$\big[E(k,-n)\big] := \big[E(k,1)\big] - \big[E(0,n+1)\big],$$

for $n \geq 0$, then the rule

$$\big[E(j,n)\big] + \big[E(k,m)\big] = \big[E(j+k, m+n)\big]$$

works for all $n \in \mathbb{Z}$, showing that $K(S^2) = \mathbb{Z} \oplus \mathbb{Z}$.

What about the ring structure?

It's clear that $[1] = \big[E(0,1)\big]$ acts as the identity. Now the isomorphism $L \oplus L \cong L^2 \oplus 1$ in $\mathrm{Vect}_2(S^2)$, shows that $[L] + [L] = [L]^2 + [1]$ in $\overline{\mathrm{Vect}}_2(S^2)$ and thus $[L]^2 = 2[L] - 1$, or $\big([L]-1\big)^2 = 0$, in $K(S^2)$.

[10] There is some fun to be had writing down an explicit homotopy. Bott's book [4] gives a general method for constructing similar homotopies, but this requires more trivial summands. So in this case it yields only $L \oplus L \oplus 1 \cong L^2 \oplus 1 \oplus 1$.

We have $[L]-[1]=\big[E(1,0)\big]$, so that the typical element of $K(S^2)$ is $\big[E(k,n)\big]=n[1]+k\big([L]-[1]\big)$. Hence we see that, as a ring,

$$K(S^2)=\mathbb{Z}[t]/(t^2),$$

where $t=[L]-[1]$.

7. Representing the functors

This simple calculation can be generalised to a huge extent. Within it are contained the germs of proofs of the K-theory Thom isomorphism and of periodicity; see the books of Atiyah, Bott and Karoubi [2, 4, 5].

But we'll go in a different direction. Our north-south clutching construction meant we were thinking of S^2 as the suspension of S^1. Exactly the same applies to any suspension. So $\overline{\mathrm{Vect}}_n(SA)=\big[A,GL(n,\mathbb{C})\big]=\big[A,U(n)\big]$.

In fact there is a space $BU(n)$ such that $\overline{\mathrm{Vect}}_n(X)=\big[X,BU(n)\big]$. More precisely there is a n-dimensional vector bundle E_n over $BU(n)$ such that every element of $\overline{\mathrm{Vect}}_n(X)$ arises as f^*E_n for some $f:X\to BU(n)$, which is unique up to homotopy.[11] For example, $BU(1)=CP^\infty$, and E_1 is a simple generalisation of the bundle L over CP^1; see §8.

The loop space $\Omega BU(n)$ is homotopy equivalent to $U(n)$. So, by adjointness, $\big[SA,BU(n)\big]=\big[A,\Omega BU(n)\big]=\big[A,U(n)\big]$, consistent with what we found before.

Computing $\overline{\mathrm{Vect}}_n(X)=\big[X,BU(n)\big]$ is generally very hard. The passage to K-theory not only means we end up with an algebraic structure we understand (a ring), but also makes the computation much more feasible. The lucky thing is that enough is preserved in this process to solve many geometrical problems.

Just as the unitary groups fit inside each other

$$U(n)\subset U(n+1)$$

and we can form their union U, we can do the same with the classifying spaces to form BU. This is a very nice space, which we can describe explicitly (but not now), and it represents K-theory. Precisely

$$K(X)=\big[X,\mathbb{Z}\times BU\big].$$

We can recalculate $K(S^2)$ as

$$\big[S^2,\mathbb{Z}\times BU\big]=\mathbb{Z}\oplus\pi_2(BU)=\mathbb{Z}\oplus\pi_1(U)=\mathbb{Z}\oplus\mathbb{Z}.$$

Now $\Omega(\mathbb{Z}\times BU)=U$, the $\mathbb{Z}$ factor has no effect—these are based loops—and it just so happens that the converse is true

$$\Omega U=\mathbb{Z}\times BU.$$

[11]Complication: $BU(n)$ is an infinite dimensional space, so we have to be careful talking about vector bundles over it.

Let me emphasis that the first of these is a formality (the standard property of classifying spaces), but the second is a very deep theorem: *Bott periodicity.* BU is a special space; apart from a factor $\mathbb{Z}$, it is its own double loop space:

$$\Omega^2 BU = \mathbb{Z} \times BU.$$

The $\mathbb{Z}$ is, of course, $\pi_2(BU)$.

There are many proofs. I rather like the one in [3], in which the homology of ΩU is calculated, shown to be the same as that of BU, and then a map is constructed which induces an isomorphism. Other proofs are much more geometrical; some work directly with bundles.

Once we have periodicity, we have a *spectrum* $\mathbf{K}$, in which the spaces are alternately $\mathbb{Z} \times BU$ and U.

And a spectrum determines a *cohomology theory.* Thus we have $K^n(X)$ for all $n \in \mathbb{Z}$, and periodicity says that $K^n(X) = K(X)$ for n even.[12]

In fact $K^*(X) = [X, \mathbf{K}]$, meaning the graded group of maps in the stable category (whose objects are spectra). In fact $K^*(X)$ is a graded ring, since $\mathbf{K}$ is a *ring spectrum.* We need to emphasise that this is when X is a *space*, since the definition of $K^*(X)$ as $[X, \mathbf{K}]$ works perfectly well when X is a spectrum.[13]

A spectrum also yields a *homology* theory $K_*(X) = \pi_*(X \wedge \mathbf{K})$. This is "dual" to K-cohomology theory in the appropriate sense. In fact when the groups are torsion-free, $K^*(X)$ and $K_*(X)$ are dual groups. There is no nice geometric construction of the K-homology of a space, but it does arise naturally in the context of C^*-algebras.

Now we have a cohomology theory we can adapt all the apparatus of ordinary cohomology theory to the K-theory context. Most of this work well, and it allows the computation of the K-theory of most reasonable spaces. This is particularly the case for spaces related to Lie groups. Sometimes the K-theory is nicer than the cohomology. For example, $K^*(G)$ is torsion-free, for a compact Lie group G, but $H^*(G)$ can have 2, 3 or 5-torsion.

We can think of K-theory as just one weapon in our arsenal of invariants for tackling homotopy problems, and it can be quite powerful.

But $K^0(X)$ can also be thought of as the first approximation to calculating $\mathrm{Vect}_n(X)$.

8. Operations and Cooperations

To tie in my talk with Reg Wood's, I will end with a lightning introduction to operations in K-theory, in which I just tell you the facts (or an approximation[14] to them) without proving anything.

[12] From now on, and perhaps already, we are being a little careless with base points.

[13] What goes wrong for a general spectrum is that there is no diagonal map. We need the diagonal to internalise the product just like in §4.

[14] From this point on my carelessness with basepoints reaches criminal proportions.

Recall that the K-functor is represented by the space $\mathbb{Z}\times BU$. It follows that natural transformations of that functor correspond to elements of $K(\mathbb{Z}\times BU) = [\mathbb{Z}\times BU, \mathbb{Z}\times BU]$.

Within the set of all natural transformations, the *additive* ones correspond to the *primitive* elements of the Hopf algebra $K(\mathbb{Z}\times BU)$. The diagonal of this Hopf algebra derives from the product $(\mathbb{Z}\times BU)\times(\mathbb{Z}\times BU)\to(\mathbb{Z}\times BU)$ which itself derives from the direct sum of bundles and so gives the addition in $K(X)$.

Now $PK(BU)\to K(BU)\to K\big(BU(1)\big)$ is an isomorphism,[15] and so additive operations correspond to elements of $K\big(BU(1)\big)$.

With the set of of additive operations, the *multiplicative* ones correspond to the *group-like* elements of the Hopf algebra $K\big(BU(1)\big)$. The comultiplication of this Hopf algebra derives from the product $BU(1)\times BU(1)\to BU(1)$, which itself derives from the tensor product of bundles and so gives, when extended to BU, the multiplication in $K(X)$.

Now $BU(1) = \mathbb{C}P^\infty$ which we can identify with the set of *monic* complex polynomials, and then the product is multiplication of polynomials.

There is a line bundle L over $\mathbb{C}P^\infty$ generalising exactly the line bundle we constructed over CP^1. We define L as a sub-bundle of the trivial (infinite-dimensional) bundle $E = \mathbb{C}P^\infty\times\mathbb{C}[x]$:

$$L = \big\{\, \big(f(x), \lambda f(x)\big) : f(x)\in\mathbb{C}[x] \text{ is monic and } \lambda\in\mathbb{C}\,\big\}.$$

If $m:\mathbb{C}P^\infty\times\mathbb{C}P^\infty\to\mathbb{C}P^\infty$ is the product $\big(f(x),g(x)\big)\mapsto f(x)g(x)$, then m^*L is the bundle

$$\big\{\, \big(f(x), g(x), \lambda f(x)g(x)\big)\,\big\}$$

over $\mathbb{C}P^\infty\times\mathbb{C}P^\infty$.

On the other hand the bundle $L\boxtimes L$ is defined inside $E\boxtimes E$, and, since $\mathbb{C}[x]\otimes\mathbb{C}[x] = \mathbb{C}[x,y]$, we see that

$$\begin{aligned} &L\boxtimes L\\ &\quad := \big\{\, \big(f(x), g(x), \lambda f(x)g(y)\big) : f(x), g(x)\in\mathbb{C}[x] \text{ are monic and } \lambda\in\mathbb{C}\,\big\}. \end{aligned}$$

Setting $y = x$ defines a vector bundle homomorphism $L\boxtimes L\to m^*L$, which is clearly[16] an isomorphism.

This shows $m^*: K(\mathbb{C}P^\infty)\to K(\mathbb{C}P^\infty\times\mathbb{C}P^\infty) = K(\mathbb{C}P^\infty)\widehat{\otimes}K(\mathbb{C}P^\infty)$ sends $[L]$ to $[L]\otimes[L]$.

But $K(\mathbb{C}P^\infty) = \mathbb{Z}[[t]]$, where $t := [L]-[1]$. So

$$m^*(t) = t\otimes 1 + 1\otimes t + t\otimes t.$$

This structure is the *multiplicative formal group law*. The group-like elements are the powers of $1+t = [L]$. Thus we get *Adams operations*: natural ring

[15]This is (one incarnation of) the *splitting principle*.

[16]What's the kernel?

homomorphisms $\Psi^k : K(X) \to K(X)$, for any integer k, corresponding to $(1+t)^k \in K\big(BU(1)\big)$.

The operation Ψ^k sends $[L]$ to $[L^{\otimes k}] = [L]^k$, in fact it acts as the k-th power on *all* line bundles since L is the *universal* line bundle, i.e., the bundle E_1 of §7. It follows that Adams operations compose as $\Psi^j \circ \Psi^k = \Psi^{jk}$.

As a purely algebraic fact, the dual of the power series ring $\mathbb{Z}[[t]]$ with the multiplicative formal group law defining the coproduct is the ring

$$\mathrm{Int}(\mathbb{Z}) = \big\{\, f(w) \in \mathbb{Q}[w] : f(\mathbb{Z}) \subseteq \mathbb{Z} \,\big\}$$

of *integer-valued polynomials*. It follows that $K_0(\mathbb{C}P^\infty) \cong \mathrm{Int}(\mathbb{Z})$. In this context Ψ^k appears as the functional $\mathrm{Int}(\mathbb{Z}) \to \mathbb{Z}$ given by $f(w) \mapsto f(k)$.

Now these are, in general, *unstable* operations. The ring[17] of *stable* operations in K-theory is $K^*(\mathbf{K}) = [\mathbf{K}, \mathbf{K}]$. This is the K-theory version of the Steenrod algebra. It's a big gadget, so Adams had the idea, probably inspired by Milnor's computation of the dual of the Steenrod algebra, of starting with $K_*(\mathbf{K}) = \pi_*(\mathbf{K} \wedge \mathbf{K})$. Remarkably perhaps,[18] $K_0(\mathbf{K})$ is not much bigger than $K_0(\mathbb{C}P^\infty)$:

$$\begin{aligned} K_0(\mathbf{K}) &= \mathrm{Int}(\mathbb{Z})[w^{-1}] \\ &= \big\{\, f(w) \in \mathbb{Q}[w, w^{-1}] : f(k) \in \mathbb{Z}\big[\tfrac{1}{k}\big] \text{ for any } k \in \mathbb{Z} \smallsetminus \{0\} \,\big\}. \end{aligned}$$

Now, if $k \neq 0$, Ψ^k extends as

$$\begin{tikzcd} \mathrm{Int}(\mathbb{Z}) \arrow[r, "\Psi^k"] \arrow[d] & \mathbb{Z} \arrow[d] \\ \mathrm{Int}(\mathbb{Z})[w^{-1}] \arrow[r, "\Psi^k"] & \mathbb{Z}[\frac{1}{k}] \end{tikzcd}$$

Conclusion: Ψ^k is not a stable operation unless $k = \pm 1$ (or unless we introduce coefficients in which k is invertible).

It is a fact that $K_0(\mathbf{K})$ is free as an abelian group; see [1]. Hence the dual $K^0(\mathbf{K})$, which is the algebra of degree zero, stable operations in K-theory, is uncountable.

A 25-year-old unsolved problem: Write down explicitly an element of $K^0(\mathbf{K})$ which is not of the form $a\Psi^1 + b\Psi^{-1}$, or, better still, describe all elements.

The best progress to date is recent joint work with Martin Crossley and Sarah Whitehouse in which the problem is solved with p-local coefficients, i.e., all primes except one are inverted. Previous workers had achieved this with p-adic coefficients.

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[17] You may object that $\mathbf{K}$ is a spectrum, not a space; see footnote 13. But this is a ring under *composition*.

[18] We might expect the procedure $\mathbb{C}P^\infty \to BU \to \mathbf{K}$ to give us a rather larger extension.

References

[1] J. F. Adams and F. W. Clarke, *Stable operations on complex K-theory*, Illinois J. Math. **21** (1977), 826–829.

[2] M. F. Atiyah, *K-theory*, W. A. Benjamin, Inc., New York-Amsterdam, 1967.

[3] R. Bott, *The space of loops on a Lie group*, Michigan Math. J. **5** (1958), 35–61.

[4] ———, *Lectures on $K(X)$*, Mathematics Lecture Note Series, W. A. Benjamin, Inc., New York-Amsterdam, 1969.

[5] M. Karoubi, *K-theory, an introduction*, Grundlehren der Mathematischen Wissenschaften, vol. 226, Springer-Verlag, Berlin, 1978.

[6] J. L. Kelley, *General Topology*, Van Nostrand, Toronto-New York-London, 1955. Reprinted as Graduate Texts in Mathematics, vol. 27, Springer-Verlag, New York-Berlin, 1975.